

SKETCHING THE INFLATON POTENTIAL

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Abstract

Based on solutions of the Stewart-Lyth inverse problem it is argued that in the CMB data analysis the parametrization of the primordial spectra from inflation must include the ‘running’ of both, scalar and tensorial, spectral indices if information beyond the exponential potential model is wanted to be detected.

Keywords: Inflation, inflaton potential, CMB.

Introduction

Usually, to understand a grownup behavior it is necessary to look back into his childhood. This seems to be also the case in Cosmology. (See Refs. [1] for reviews and references on the topics mentioned in this introduction).

The eldest picture we have of our universe comes from the times when it was about 10^5 years old. In that epoch matter became cold enough for radiation to decouple and almost freely travel across the space up to the present. Since cosmic time ticks logarithmically and the current age of the universe is estimated to be about 10^{10} years, this nearly uniform cosmic background of radiation is a tidy picture of the young universe.

Nuclear physics allow us to trace the cosmic history back to the times when the lightest chemical elements were synthesized. The predicted abundances of these elements can be matched with current surveys, providing evidence about the universe when it was a three seconds old kid.

High-energies physics as described by the Standard Model of Particles allow us to glance a little bit further into the past, but the description of the events happening immediately after the birth of our universe still remains highly speculative. The main problem here is the lack of a consistent and tested theory, and the practical impossibility of reproducing

the relevant events in laboratory conditions because of the very high energies involved. It is, therefore, quite stimulating that the inflationary paradigm links the physics in the very early universe to the current cosmological state.

A period of rapid accelerated expansion of the universe just after its birth can explain several features of the currently observed universe like its age, its size and its topology. Perhaps the more attractive feature that can be explained in this framework are the initial conditions for our own existence. In a perfectly homogeneous universe there are not chances for such complex structures like we are to arise. At the epoch when non-relativistic matter dominated over the relativistic one (or radiation), inhomogeneities acting like perturbations in the gravitational potential were required to seed the formation of galaxies through gravitational instability. Ultimately, this process led to the formation of solar systems and planets like the Earth. Those perturbations produced when the pressure of the radiation was still high enough to compensate gravity attraction induced an oscillatory motion of expansions and contractions. This motion led to inhomogeneities in the background temperature which we observe today as anisotropies in the cosmic microwave background (CMB) radiation. Therefore, to describe the formation of galaxies and the CMB anisotropies spectrum it is necessary to set the spectrum of the corresponding initial perturbations of the gravitational potential. Given the very special characteristics of this primordial spectrum, the inflationary scenario is the most widely accepted mechanism for setting these initial conditions [1, 2]. In an inflationary universe quantum fluctuations of matter and space-time are stretched by the expansion up to cosmological scales well beyond the distance within which causal interaction can take place. After the end of inflation these fluctuations reenter the causal horizon producing perturbations in the gravitational potential.

It is remarkable that the kind of primordial perturbation spectra generated in the simplest version of the inflationary scenario, the single scalar field scenario, fits quite well as the seeds for the CMB anisotropies [1, 2]. In this scenario the expansion is driven by the potential energy of the, so called, inflaton field. This way, if it is assumed, as we do here, that the inflaton physics closely correspond to the actual scenario of the very early universe, then recalling the earliest memories of our universe is equivalent to sketching the inflationary potential.

The aim of this contribution is to discuss how the Stewart-Lyth inverse problem [3] can be used as a powerful tool for drawing a photo-robot of the inflaton potential.

1. Stewart-Lyth inverse problem

To describe the inflationary dynamics and the corresponding perturbations it has proved to be useful to introduce the set of horizon flow functions [4]:

$$\epsilon_0 \equiv \frac{d_H(N)}{d_{\text{Hi}}}, \quad (1)$$

and,

$$\epsilon_{m+1} \equiv \frac{d \ln |\epsilon_m|}{dN}, \quad m \geq 0. \quad (2)$$

where $d_H \equiv 1/H \equiv a/\dot{a}$ denotes the Hubble distance, with d_{Hi} evaluated at some initial time t_i and dot stands for differentiation with respect to cosmic time. The scale factor a measures the expansion of the spatial volume, and $N \equiv \ln(a/a_i)$ is the e-folds number. The first horizon flow function ϵ_1 can be written in several useful ways:

$$\epsilon_1 \equiv \frac{d \ln d_H}{dN} = \dot{d}_H = \frac{3}{2} \frac{\rho + p}{\rho} = 3 \frac{T}{T + V} = \kappa \frac{T}{H^2}, \quad (3)$$

where ρ and p are, respectively, the energy density and the pressure in a universe dominated by the potential energy, $V(\phi)$, of the inflaton field, ϕ . $T \equiv \dot{\phi}^2/2$, $\kappa = 8\pi/m_{\text{Pl}}^2$ is the Einstein constant and m_{Pl} is the Planck mass. Inflation happens for $0 \leq \epsilon_1 < 1$. For $m > 1$, ϵ_m may take any real value.

At the high energies corresponding to the inflationary period, the inflaton energy density and the space-time metrics undergo quantum fluctuations. These are the fluctuations that stretched by the expansion reentered the Hubble horizon, d_H , close to the point of equal density of matter and radiation, to seed the formation of large scale structure and the CMB anisotropies. The spectra of these seeds can be parametrized by the series:

$$\ln A_S^2(k) = \ln A_S^2(k_*) + \Delta(k_*) \ln \frac{k}{k_*} + \frac{1}{2} \frac{d\Delta(k)}{d \ln k} \Big|_{k=k_*} \ln^2 \frac{k}{k_*} + \dots \quad (4)$$

$$\ln A_T^2(k) = \ln A_T^2(k_*) + \delta(k_*) \ln \frac{k}{k_*} + \frac{1}{2} \frac{d\delta(k)}{d \ln k} \Big|_{k=k_*} \ln^2 \frac{k}{k_*} + \dots, \quad (5)$$

where A_S and A_T stand respectively for the normalized amplitudes of the density (scalar) and metrics (tensor) perturbations, and k_* is the wavenumber corresponding to a pivotal length scale. Functions $\Delta(k)$ and $\delta(k)$ will be called here the scalar and tensorial spectral indices respectively. Their derivatives with respect to $\ln k$ are known as the ‘running’ of the spectral indices. In the procedure of fitting the CMB data the order where series (4) and (5) are truncated is determined by

the precision of the observations. It is worthy to mention that a careful analysis of the perturbations generated during inflation shows that the spectra of such perturbations while reentering the Hubble horizon are generically almost scale-invariant (see Ref. [1] for details). With this regard, one can assume that each higher term in series (4) and (5) is smaller than the corresponding lower order term.

Typically, most of CMB data analyses have neglected the possible effects of the primordial gravitational wave spectrum [2]. The tensor contribution to the CMB spectrum can be parametrized in terms of the quantity

$$r \equiv \alpha \frac{A_T^2}{A_S^2}, \quad (6)$$

representing the relative amplitudes of the tensor and scalar perturbations, where the constant, α , depends on the particular normalization of the spectral amplitudes that is chosen. In the last few years, however, there has been a growing recognition that the role of the tensor perturbations deserves more attention when determining the best-fit values of the cosmological parameters (For a recent review, see, e.g., Ref. [2]).

The Stewart-Lyth inverse problem (SLIP) was introduced in Ref. [3] as a method for finding the inflaton potential using information on the functional form of the spectral indices. With this aim, the following non-linear differential equations must be solved:

$$2C\epsilon_1\hat{\epsilon}_1 - (2C + 3)\epsilon_1\hat{\epsilon}_1 - \hat{\epsilon}_1 + \epsilon_1^2 + \epsilon_1 + \Delta = 0, \quad (7)$$

$$2(C + 1)\epsilon_1\hat{\epsilon}_1 - \epsilon_1^2 - \epsilon_1 - \delta = 0, \quad (8)$$

where $C = -0.7293$ and a circumflex accent denotes differentiation with respect to $\tau \equiv \ln H^2$. These equations are derived from next-to-leading order expressions for the spectral indices in terms of the horizon flow functions (2) [3]. In Ref. [5] it was shown that to this order $\delta \leq 0$. The validity of the conclusions to be drawn here are constrained by the next-to-leading order precision.

Then, having an expression for $\tau(\epsilon_1)$ (which is typically the form of the solutions to Eqs. (7) and (8)) the corresponding inflaton potential is given by the parametric function [5],

$$V(\phi) = \begin{cases} \phi(\epsilon_1), \\ V(\epsilon_1), \end{cases} \quad (9)$$

where,

$$V(\epsilon_1) = \frac{1}{\kappa} (3 - \epsilon_1) \exp [\tau(\epsilon_1)] , \quad (10)$$

and,

$$\phi(\epsilon_1) = -\frac{2(C+1)}{\sqrt{2\kappa}} \int \frac{\sqrt{\epsilon_1} d\epsilon_1}{\epsilon_1^2 + \epsilon_1 + \delta} + \phi_0. \quad (11)$$

Here V_0 and ϕ_0 are integration constants. The SLIP solutions are constrained by the following conditions [5]:

$$\begin{cases} \hat{\epsilon}_1 \frac{d\phi}{d\epsilon_1} < 0, \\ \hat{\epsilon}_1 \frac{dV}{d\epsilon_1} > 0. \end{cases} \quad (12)$$

In order to link the SLIP solution with the primordial spectra given by the series (4) and (5) it proved to be useful to rewrite Eqs. (7) and (8) by converting from derivatives with respect to τ to derivatives with respect to $\ln k$, with k corresponding to the wavelength crossing the Hubble horizon by the first time, i. e., $k = aH$ [6].

2. Constraining the inflaton potential

The Stewart-Lyth inverse problem has two strong drawbacks. First, the full power of this procedure can be used only when information on the functional forms of both spectral indices is available. Unfortunately, this information is rather difficult to be directly obtained from observations. In addition, simple functional forms of the spectral indices involve great difficulties while solving the SLIP. In spite of these drawbacks, there is an alternative way of using the SLIP related expressions. This method allows to test for the internal consistency in the procedure of fitting the CMB data by finding and describing the inflaton potential corresponding to the assumptions on the primordial perturbations used in that procedure. Thus, the SLIP can help in constraining the possible inflaton potentials by linking distinct features of the power spectra to the inflationary dynamics.

Historically, the first analytical calculations of the form of the initial conditions required for galaxy formation due to Harrison and Zel'dovich (see Ref. [1] for details) yield scale-invariant spectra, i.e., constant amplitudes ($\Delta = \delta = 0$). Current precision of the CMB measurements still allows for this kind of spectra to be used as initial conditions [2]. However, in the inflationary scenario some degree of scale dependence is necessarily present if a dynamical mechanism for ending the accelerated expansion acted in the very early universe in order to recover the success of the Hot Big Bang Theory as a sequence of a radiation and a matter dominated universes.

Since the contribution of the tensor modes to the CMB anisotropies is typically very small, one can wonder whether the actual inflaton po-

tential could produce perturbations spectra where the scale dependence is entirely contained in the scalar component, i. e., $\delta = 0$ and $\Delta = \Delta(k)$.

The corresponding SLIP was solved in Ref. [5]. The inflaton potential has the form,

$$V(\phi) = V_0 \frac{3 - \tan^2 \left[\frac{\sqrt{2\kappa}}{4(C+1)} (\phi - \phi_0) \right]}{\cos^{4(C+1)} \left[\frac{\sqrt{2\kappa}}{4(C+1)} (\phi - \phi_0) \right]}. \quad (13)$$

Nevertheless, the analysis of the behavior of the scalar index,

$$\begin{aligned} \Delta(k) = & \frac{1}{8(C+1)^2} \left\{ \left(\frac{k}{k_0} \right)^{1/(C+1)} - \sqrt{\left(\frac{k}{k_0} \right)^{1/(C+1)} \left[\left(\frac{k}{k_0} \right)^{1/(C+1)} - 4 \right]} \right\} \\ & \times \left\{ \left(\frac{k}{k_0} \right)^{1/(C+1)} + 2C - \sqrt{\left(\frac{k}{k_0} \right)^{1/(C+1)} \left[\left(\frac{k}{k_0} \right)^{1/(C+1)} - 4 \right]} \right\}, \quad (14) \end{aligned}$$

corresponding to the potential (13), shows that, for large k , Eq. (14) converges to $\Delta = 1/2(C+1) \approx 1.85$, which is too far from values allowed by theory and observations [1, 2].

Therefore, a correct parametrization of the inflationary spectra must take into account terms beyond the constant ones in both series (4) and (5). The next step is, then, to consider spectra slightly tilted from scale invariance, i.e., with constant and close to zero spectral indices. In that case the spectra take on a power-law form. It is well known that an exponential potential produces exactly such perturbation spectra. This scenario is called power-law inflation because $a \propto t^p$, where $p \gg 1$ is a constant [7]. Using Eqs. (3), it is easy to check that for this model, $\epsilon_1 = 1/p$.

We proved in Ref. [5] that, even to next-to-leading order, the only SLIP solution corresponding to these conditions on both spectra, tensorial and scalar, is power-law inflation. If, in the same mood as it was done while analyzing the case $\delta = 0$, the condition of a power-law spectrum is relaxed by allowing scale dependence only for the scalar index, then it will be obtained that, though power-law inflation is still a trivial solution, other SLIP solutions arise [5]. (Hereafter the involved formulas for the SLIP solutions are omitted. For details see the cited references). These solutions converge to power-law inflation; the scalar index converges to a constant realistic value from above or from below, depending on the initial conditions. Since the depart from power-law behavior takes place at large angular scales where the cosmic variance is dominant, then it will be very difficult from the observational point of view to distinguish between these SLIP solutions and power-law inflation.

This way, if one wants to move beyond the power-law scenario, it is necessary to include the running of both spectral indices in the parametrization given by Eqs. (4) and (5). It is reasonable, for instance, to consider that the scale dependence of the tensorial spectral index is distinctly perceived only up to next-to-leading order in terms of ϵ_1 , consistently with the approximation used to derive Eqs. (7) and (8). We solved the SLIP with the ansatz [8],

$$\delta(\epsilon_1) = - \left[(1+a)\epsilon_1^2 + (1+b)\epsilon_1 + c \right], \quad (15)$$

where a, b, c are real numbers. It was found that for $b^2 > 4ac$, power-law inflation is again an attractor of the corresponding inflationary dynamics. For $b^2 < 4ac$, power-law inflation is no longer an attractor but a transient regime of the dynamics. Obviously, the spectra depart from a power law before and after the quasi power-law behavior. Those perturbations produced before the quasi power-law regime are imprinted in the very large angular scales, being the detection of their signatures difficult because of the cosmic variance. On the other hand, since inflation ends very fast after the quasi power-law regime is left behind, the scales crossing the horizon at that time were extraordinarily small and reentered it back immediately with not relevant effect. If the actual inflaton potential is close to this model, it will be also very difficult to observe any difference from the exponential potential. Then, even including the running of both spectral indices in the parametrization of the primordial spectra could be not sufficient to get ride of the power-law bias.

There are cases where power-law inflation is a repeller of the inflationary dynamics. In Ref. [9] the SLIP was solved with the condition of constant tensor to scalar ratio, r . This was partially motivated by the possibility, in the near future, of estimating a constant central value for r from the observation of the CMB polarization [1, 2]. From Eqs. (4), (5), (7) and (8), it is simple to show that the condition $r = \text{constant}$ is equivalent to $\Delta(k) = \delta(k)$. Using this, after adding Eqs. (7) and (8),

$$2C\epsilon_1\hat{\epsilon}_1 - (\epsilon_1 + 1)\hat{\epsilon}_1 = 0. \quad (16)$$

Obviously, a trivial solution for this case is $\epsilon_1 = 1/p = \text{constant}$, corresponding to power-law inflation. The remaining SLIP solutions depart very quickly from this power-law solution. In one case δ grows and becomes positive, thus indicating a breakdown in the next-to-leading order analysis. For the other case, δ begins to evolve extremely rapidly, probably indicating that the running of the spectral index, $d\Delta/d\ln k$, becomes too large. Either way, observational constraints are difficult to satisfy. Thus, the potential in the region open to observation must be sufficiently close to the exponential (power-law inflation) model.

3. Conclusions

The quality of the photo-robot of the inflaton potential depends crucially on the order of the series used to parametrize the primordial spectra while fitting the CMB anisotropies spectrum. In turn, that order depends on the precision of the available observational data.

If the inflaton potential is suspected to differ from the exponential potential corresponding to power-law inflation, then quadratic terms of both scalar and tensor modes are necessary (but perhaps not sufficient) to observe the differences.

If, while fitting the CMB data, the running of both spectral indices happen to be distinctly different from zero, then the constant value of the tensor to scalar ratio as determined from the CMB polarization likely will not be characteristic of a wide range of scales.

When this contribution was ready to be submitted, a paper by Leach et al. [10] was posted at Los Alamos including a very interesting discussion on the parametrization of the primordial spectra from inflation.

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